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LOW-ANGLE DISCLINATIONS ON ELASTIC PLANE: A GAUGE-THEORY APPROACH

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A gauge model of disclinations in elastic planar media is studied within the linear approximation. It is shown that an exact vortex-like solution for a straight wedge disclination does not depend on the coupling constants of the theory.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Малоугловые дисклинации на упругой плоскости: калибровочный подход

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В рамках линейного приближения изучена калибровочная модель дисклинаций на упругой плоскости. Показано, что точное вихревое решение для прямолинейной клиновой дисклинации не зависит от констант связи теории.

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One of the modern trends in condensed matter physics is a study of materials taking the form of elastic surfaces (fullerenes, carbon nanotubes, membranes) [1, 2]. An important role in these objects play topological defects, first of all disclinations. As is known, there are always twelve disclinations on the closed hexatic elastic surface due to the Euler theorem. For this reason, the disclination-induced effects on elastic surfaces are of considerable interest.

As has been shown recently, an appropriate model for the description of disclinations in elastic materials is the Edelen-Kadic (EK) gauge model [3]. In particular, within this model an exact solution for a topologically unstable disclination vortex was found in [4]. It was shown that the strain and stress fields caused by this vortex coincide with those for the straight wedge disclination in a classical theory of disclinations. This finding confirms the view of a disclination as the vortex of an elastic medium. Thus disclinations are among the other known vortex-like objects in various media. It is interesting to note that the elastic flux due to disclinations was found to be completely determined by the gauge vector fields.

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Notice that the above-mentioned solution was obtained within the restricted model when only rotational symmetry was taken into account. In addition, the dislocation-induced contribution to the Lagrangian was omitted. As a matter of fact, this contribution always exists (so-called disclination driven dislocations [3]). The goal of the present paper is to consider the most general model for disclinations on the elastic plane involving all possible terms.

Let us start from the Lagrangian that is invariant under the inhomogeneous action of the gauge group S0(2) [5]

$$L = L_\chi + L_\phi + L_W, \tag{1}$$

where

$$L_\chi = \frac{1}{2}\rho_0 B_3^i \delta_{ij} B_3^j - \frac{1}{8}[\lambda (E_{AB}\delta^{AB})^2 + 2\mu (E_{AB}\delta^{AC}\delta^{BD}E_{CD})] \tag{2}$$

describes the elastic properties of the material while

$$L_\phi = -\frac{1}{2}s_1 \delta_{ij} D^i_{ab} k^{ac} k^{bd} D^j_{cd}, \tag{3}$$

$$L_W = -\frac{1}{2}s_2 F_{ab} g^{ac} g^{bd} F_{cd} \tag{4}$$

describe a disclination-induced contribution. In comparison with the general gauge model [3] which considers both the dislocations and disclinations, we omit here the fields due to dislocations. However, there is a contribution L_ϕ that comes from the dislocation part of the general Lagrangian [5]. As is known, there are no pure disclinated materials and, in fact, disclinations always give rise to disclination driven dislocations. In (2—4) $E_{AB} = B^i_A \delta_{ij} B^j_B - \delta_{AB}$ is the strain tensor, $D^i_{ab} = \varepsilon^i_j F_{ab}\chi^j$, $F_{ab} = \partial_a W_b - \partial_b W_a$, and s_1 and s_2 are the coupling constants. In accordance with the minimal replacement arguments, we have

$$B^i_a = \partial_a \chi^i + \varepsilon^i_j \chi^j W_a, \tag{5}$$

where $\chi^i(X^a) = \chi^i(X^A, T)$ characterizes the configuration at time T in terms of the coordinate cover (X^A) of a reference configuration, W_a is the compensating gauge field associated with the disclination field. In (3) the quantities k_{ab} are given by $k^{AB} = -\delta^{AB}$, $k^{33} = 1/y$, and $k^{ab} = 0$ for $a \neq b$, whereas in (4) $g^{AB} = -\delta^{AB}$, $g^{33} = 1/\xi$, and $g^{ab} = 0$ for $a \neq b$. The parameters y and ξ are the two positive «propagation parameters», ε^i_j is a completely antisymmetric tensor, $\varepsilon^1_2 = 1$, and λ and μ are the Lamé constants. We have used here the same notation as in [3,5]. The Euler-Lagrange equations for (1) take the following form in the static case

$$\partial_A \sigma_k^A = \varepsilon_k^j W_C \sigma_j^C + s_1 \chi^l \delta_{lk} F_{AB} F^{AB}, \tag{6}$$

$$\sigma_A^j \delta_{ij} \varepsilon_l^i \chi^l = 2\partial_B [(s_1(\chi)^2 + s_2) F^{BA}], \tag{7}$$

where $(\chi)^2 = \chi^l \chi_l$, $l = 1,2$. To avoid cumbersome expressions we will sometimes omit the right order of the top and bottom indices which can be easily restored by using the appropriate δ-symbols. The stress tensor σ_c^j is determined to be

$$\sigma_C^j = \frac{1}{2} [\lambda (E_{AB} \delta^{AB}) B_C^j + 2\mu (E_{CB} B_B^j)]. \tag{8}$$

First, let us introduce the dimensionless variables via $x^i = \sqrt{s_2/s_1}\, \tilde{x}^i$ and $W_A = \sqrt{s_2/s_1}\, \tilde{W}_A$. The Euler-Lagrange equations (6) and (7) become

$$\tilde{\partial}_A \tilde{\sigma}_k^A = \varepsilon_k^j \tilde{W}_C \tilde{\sigma}_j^C + \frac{s_1^2}{\mu s_2} \tilde{\chi}^l \delta_{lk} \tilde{F}_{AB} \tilde{F}_{AB}, \tag{9}$$

$$\tilde{\sigma}_A^j \delta_{ij} \varepsilon_l^i \tilde{\chi}^l = \frac{2 s_1^2}{\mu s_2} \tilde{\partial}_B [((\tilde{\chi})^2 + 1) \tilde{F}^{BA}], \tag{10}$$

where $\chi^l(x^B) = \sqrt{s_2/s_1}\, \tilde{\chi}^l(\tilde{x}^B)$, $\partial_B = \sqrt{s_1/s_2}\, \tilde{\partial}_B$, $\tilde{F}_{AB} = \tilde{\partial}_A \tilde{W}_B - \tilde{\partial}_B \tilde{W}_A$ and the stress tensor is found to be

$$\tilde{\sigma}_j^A = \frac{K}{2} (\tilde{E}_{CD} \delta_{CD}) \tilde{B}_A^j + \tilde{E}_{AC} \tilde{B}_C^j. \tag{11}$$

Here $K = \lambda/\mu$, and the strain tensor takes the form

$$\tilde{E}_{AB} = \tilde{B}_A^i \delta_{ij} \tilde{B}_B^j - \delta_{AB} \tag{12}$$

with $\tilde{B}_A^i = \tilde{\partial}_A \tilde{\chi}^i + \varepsilon_j^i \tilde{\chi}^j \tilde{W}_A$. To simplify notation, we will omit the symbol «tilde» below. The coupled nonlinear field equations (9) and (10) are difficult to solve in the general case. Usually, the linearization procedure is used and the displacement vector u^i is introduced as follows

$$\chi^i(x^B) = \delta_a^i x^a + u^i(x^B). \tag{13}$$

Then, with the scaling parameter ε all the fields are expanded in series of ε:

$$u^i = \varepsilon u_1^i + \varepsilon^2 u_2^i + \ldots, \quad W_A = \varepsilon W_{1A} + \varepsilon^2 W_{2A} + \ldots \tag{14}$$

Taking into account only disclination-induced displacements u_1^i that are of interest here, we get the first order equations in the following form,

$$\partial_B [(x^2 + y^2 + 1)\, F^{BA}] = 0, \tag{15}$$

$$\Delta \mathbf{u} + (K+1)\, \nabla\, \mathrm{div}\, \mathbf{u} = \mathbf{j} + \varkappa \frac{s_1^2}{\mu s_2} F_{AB} F^{AB} \tag{16}$$

where

$$j_x = (K-1) W_y - (K+2)\, y\, \partial_x W_x + K x\, \partial_x W_y - y\, \partial_y W_y + x\, \partial_y W_x, \tag{17}$$

$$j_y = -(K-1) W_x + (K+2)\, x\, \partial_y W_y - K y\, \partial_y W_x + x\, \partial_x W_x - y\, \partial_x W_y. \tag{18}$$

Hereafter we omit the index 1 denoting the order of the approximation. Let us emphasize that we assume here $s_1^2/\mu s_2 \sim 1/\varepsilon$. Other two possibilities $s_1^2/\mu s_2 \sim \varepsilon$ and $s_1^2/\mu s_2 \sim 1$ lead to the standard theory [4]. Notice that for $s_1 = 0$ a solution of these equations was found in [4]. An interesting property of this solution was its independence on the parameter s_2 as well. This can be seen directly from (15) and (16). Let us choose the following ansatz for (15) and (16)

$$W_x = -\frac{y}{r^2} W(r), \quad W_y = \frac{x}{r^2} W(r), \quad u_i = x_i G(r), \tag{19}$$

where $r^2 = x^2 + y^2$. Using (19) we can rewrite (15) and (16) as follows

$$\partial_r \left[(r^2 + 1) \frac{W'(r)}{r} \right] = 0, \tag{20}$$

$$(K+2) \left(G''(r) + \frac{3G'(r)}{r} \right) = K \frac{W'(r)}{r} - \frac{2W(r)}{r^2} + \frac{2s_1^2}{\mu s_2} \left(\frac{W'(r)}{r} \right)^2, \tag{21}$$

where G' stands for dG/dr and W' for dW/dr. A solution of (20) takes the form

$$W(r) = C_1 \ln (r^2 + 1) + C_0. \tag{22}$$

Notice that the quantization rule for a given defect configuration reads as a circuital integral

$$\frac{1}{2\pi} \oint \mathbf{W}\, d\,\mathbf{r} = \nu. \tag{23}$$

Taking this into account, we immediately get that $C_1 = 0$. Thus the constant C_0 turns out to be in fact a topological characteristic of the defect that is the Frank index ν. For $W(r) = \nu$ (21) becomes remarkably simpler and has a solution

$$G(r) = -\frac{\nu}{K+2} \ln r - \frac{1}{2} C_2 r^{-2} + C_3. \qquad (24)$$

Since the boundary condition yields $u^i(0) = 0$ we must put $C_2 = 0$. Turning back to the dimensional variables, one finally obtains

$$u^i = -x^i \left(\frac{\nu}{K+2} \ln \sqrt{\frac{s_1}{s_2}}\, r + C_3 \right), \qquad (25)$$

where C_3 is still an arbitrary constant. As is seen, the term with s_1 and s_2 only renormalizes the constant C_3. In particular, for the straight wedge disclination on a disk of radius R with a boundary condition in the form $u^i(R) = 0$ one obtains

$$u^i = -x^i \frac{\nu}{K+2} \ln \frac{r}{R}. \qquad (26)$$

We see that parameters s_1 and s_2 drop out from (25). Similarly, for the most-used boundary condition $\sigma_{kl} n_l = 0$ at the free surfaces one can reproduce the well-known stress fields for a wedge disclination on a disk (see details in [4]). Thus one can conclude that the information which is carried out by the coupling constants s_1 and s_2 is lost within the linear approximation. Since one can expect that just a combination of these constants determines the core radius of disclinations, it is important to study the nonlinear equations (6) and (7). The results of this investigation will be published elsewhere. It is easy to check that the modified constant C_3 disappears for dipoles of disclinations as well. Indeed, the dipole consists of two disclinations with opposite directed Frank vectors. In this case, the terms with s_1, s_2, and C_3 in (25) cancel each other thus leading to standard expressions.

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